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(6) ON THE CALCULATION OF THRUST COEFFICIENT .

by

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SUMMARY

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The thrust of a rocket motor is generally calculated from the stagnation pressure, the throat area, and a non-dimensional thrust coefficient dependent upon the nozzle expansion area ratio. A comparison is made between momentum balance and pressure integral methods of calculating the ideal vacuum thrust coefficient, and the results used to examine how corrections should be made for nozzle divergence, skin friction and two-phase flow to obtain the real thrust coefficient of the motor.

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CONTENTS

1 INTRODUCTION

The thrust of a rocket motor is generally calculated in terms of the stagnation pressure (p_o) and the throat area (A_*) from the formula

$$F = C_f \, p_o \, A_*$$

where C_f is the thrust coefficient. The usual procedure is to calculate the "ideal" thrust coefficient from one-dimensional isentropic flow theory, and then to apply corrections for ambient pressure and the various nozzle "losses".

The "ideal" thrust coefficient is normally calculated by considering the overall momentum balance of the motor. It may also be calculated by considering the pressure distribution over the internal surfaces of the motor. In this memorandum the two methods are first compared, and then the various corrections are identified and their effect on the final value of the thrust coefficient is considered.

2 IDEAL THRUST COEFFICIENT

2.1 One-dimensional isentropic flow

The fundamental equation for an ideal, one-dimensional, compressible flow is Bernoulli's equation which describes the conservation of energy:

$$\frac{u^2}{2} + \frac{a^2}{\gamma-1} = \frac{a_o^2}{\gamma-1} \tag{2.1}$$

where u is the local velocity, a is the speed of sound, and a_o is the speed of sound at stagnation conditions. If we remember that

$$a^2 = \gamma RT$$

then this can be transformed to

$$\frac{T_o}{T} = 1 + \frac{\gamma-1}{2} M^2 \tag{2.2}$$

and with the gas law and isentropic flow relations

$$p = \rho RT \quad \text{and} \quad p\rho^{-\gamma} = \text{const}$$

$$\frac{\rho_o}{\rho} = \left[1 + \frac{\gamma-1}{2} M^2 \right]^{\frac{1}{\gamma-1}} \tag{2.3}$$

and

$$\frac{p_o}{p} = \left[1 + \frac{\gamma-1}{2} M^2 \right]^{\frac{\gamma}{\gamma-1}} . \tag{2.4}$$

The mass-flow rate at any section is constant:

$$\dot{m} = \rho A u = \rho_* A_* u_* \tag{2.5}$$

where a $*$ subscript denotes the throat conditions

$$u_* = a_* \qquad M_* = 1 .$$

The mass-flow rate can therefore be calculated by substituting (2.1) and (2.3) in (2.5) to give

$$\dot{m} = \frac{p_o A_*}{a_o} \cdot \gamma \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{2(\gamma-1)}} . \tag{2.6}$$

Finally, we can use (2.5) to give relationship between Mach number and area ratio;

$$\frac{A}{A_*} = \frac{1}{M} \left[1 + \frac{\gamma-1}{2} M^2 \right]^{\frac{\gamma+1}{2(\gamma-1)}} \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{2(\gamma-1)}} . \tag{2.7}$$

2.2 Momentum balance equation

If we consider the overall momentum balance of the control volume shown in Fig. 1:

$$F = \dot{m} u_e + p_e A_e$$

where u_e is the gas velocity at exit, p_e the static gas pressure and A_e the exit cross-sectional area. For the moment we shall assume that the ambient pressure is zero.

From (2.1)

$$\left(\frac{u_e}{a_o} \right)^2 = \frac{2}{\gamma-1} \left(1 - \frac{a^2}{a_o^2} \right)$$

$$= \frac{M_e^2}{\left[1 + \frac{\gamma-1}{2} M_e^2 \right]} \tag{2.8}$$

where M_e denotes the Mach number at the exit plane.

Also, combining (2.4) and (2.7):

$$\frac{p_e A_e}{p_o A_*} = \frac{1}{M_e} \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \left[1 + \frac{\gamma-1}{2} M_e^2 \right]^{-\frac{1}{2}} \tag{2.9}$$

so that, using (2.6), (2.8) and (2.9)

$$F = p_o A_* \left\{ \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \left[1 + \frac{\gamma-1}{2} M_e^2 \right]^{-\frac{1}{2}} \left[\gamma M_e + \frac{1}{M_e} \right] \right\}$$

or

$$C_{fv} = \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{(1 + \gamma M_e^2)}{M_e \left(1 + \frac{\gamma-1}{2} M_e^2 \right)^{\frac{1}{2}}} \tag{2.10}$$

2.3 Pressure integral method

While the method shown above provides a simple means of calculating the ideal thrust coefficient, it leaves some ambiguities, particularly when we come to consider how various "losses" in the thrust coefficient arise. Most

immediately we observe that the final term $(p_e A_e)$ in the momentum equation
does not represent a force acting on the physical structure of the nozzle at
all, but is really an "adjustment" made to the equation because the control
volume has been drawn across the exit plane of the nozzle. The influence of
the nozzle will actually extend downstream of this point within an envelope
bounded by the Mach lines extending from the lip of the nozzle to the centre
line. In part this ambiguity arises because the "one-dimensional flow" assump-
tion cannot be true if the flow is to expand.

The alternative approach is to observe that the thrust is generated by the
integral of the internal pressure over the surface of the nozzle:

$$F = \int p\,dA$$

where A is the axial cross-section of the flow. In fact the integration will
have to be made in two parts - one over the head-end of the motor, and the
second over the length of the nozzle. At this point we can also allow for the
finite contraction ratio of the nozzle

$$F = p_o A_o + \int_{A_o}^{A_e} p\,dA \quad . \tag{2.11}$$

If we differentiate the mass-flow equation (2.9) we have

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{du}{u} = 0 \tag{2.12}$$

and since the differential forms of the gas law and isentropic flow equations
are

$$\frac{dp}{p} - \frac{d\rho}{\rho} - \frac{dT}{T} = 0$$

and

$$\frac{dp}{p} - \gamma \frac{d\rho}{\rho} = 0$$

$$\frac{d\rho}{\rho} = \frac{1}{(\gamma-1)} \frac{dT}{T}$$

which, by differentiating (2.1) becomes

$$\frac{d\rho}{\rho} = -\frac{u \; du}{a^2} = -M^2 \frac{du}{u} \quad .$$

Equation (2.12) can therefore be written as

$$\frac{dA}{A} = (M^2 - 1) \frac{du}{u} \tag{2.13}$$

which also indicates how the nozzle must change from a contraction to an expansion as the Mach number passes unity to maintain a positive acceleration.

In fact we can use equation (2.1) to convert (2.13) entirely to terms of Mach number:

$$\frac{dM}{M} = \frac{du}{u} - \frac{da}{a}$$

and

$$u \; du + \frac{2}{\gamma-1} \cdot a \; da = 0$$

$$\therefore \quad \frac{dM}{M} = \frac{du}{u} \left[1 + \frac{\gamma-1}{2} M^2 \right]$$

and

$$\frac{dA}{A} = \frac{M^2 - 1}{\left[1 + \frac{\gamma-1}{2} M^2 \right]} \frac{dM}{M} \tag{2.14}$$

which is the differential form of (2.7). We can now substitute this equation together with (2.7) and (2.4) in (2.11):

$$F = p_0 A_0 + p_0 A_* \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \int_{M_0}^{M_e} \frac{(M^2-1)}{M^2} \left[1 + \frac{\gamma-1}{2} M^2 \right]^{-3/2} dM$$

9

or

$$C_f = \frac{A_o}{A_*} + \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \int_{M_o}^{M_e} \frac{(M^2-1)}{M^2}\left[1 + \frac{\gamma-1}{2}M^2\right]^{-3/2} dM \quad .$$

The ratio A_o/A_* will be determined by the entry Mach number to the nozzle, as in equation (2.7). When the entry Mach number is small, this term cancels with the first term in the integral, as might be expected since C_f is not very sensitive to entry Mach number. The integral then proves to be identical with (2.10)

$$C_f = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{(1 + \gamma M_e^2)}{M_e\left[1 + \frac{\gamma-1}{2}M_e^2\right]^{\frac{1}{2}}} \quad .$$

3   CORRECTION FOR AMBIENT PRESSURE

So far we have assumed that the ambient pressure is zero. In practice with a finite ambient pressure (p_a) we should continue the integral in section 2.3 over the outside of the motor. The integral is simplified by the fact that p_a may be assumed everywhere constant. The thrust is then given by

$$T = C_f p_o A_* + \int_{A_e}^{o} p_a \, dA$$

$$= C_f p_o A_* - p_a A_e \quad .$$

The integral here goes from the lip of the exit cone to the centre of the head end of the motor. Thus

$$C_f = C_{fv} - \frac{p_a A_e}{p_o A_*} \quad .$$

If p_a is not constant over the outside of the motor, e.g. due to induced flow past the motor, then it will be necessary to re-evaluate the integral to allow for this effect.

4   THRUST COEFFICIENT LOSSES

4.1   General

There are a number of losses to the overall thrust which do not affect the thrust coefficient, and we shall not consider those here. Under this category we shall ignore effects which alter the flow area of the throat or the upstream stagnation conditions. The major losses to thrust coefficient are then

- divergence losses
- skin friction losses
- two-phase flow losses

We must now consider how these losses are to be integrated into calculation of the thrust coefficient.

4.2   Divergence losses

On exit from the nozzle the flow has a component of momentum in the radial as well as the axial direction. Malina [1] calculated the loss of thrust to radial momentum by assuming that the exit flow formed a cone of streamlines originated from a point source near the throat. He showed that the reduction in axial momentum in this instance was by a factor

$$\eta_d = \frac{1 + \cos\theta}{2}$$

where θ is the half angle of the cone. Nowadays more complete computer flow predictions are available, but it appears that Malina's formula provides a high degree of accuracy if the half-angle of the flow at exit is used.

Using Malina's formula one might assume that only the velocity terms in the momentum balance equation would be affected. However, Landsbaum [2] pointed out that the exit area involved in Malina's assumption is actually the cap of a sphere and not the flat exit plane, and that the "one dimensional" calculation of C_f is initially high, and using Malina's correction on all the terms in C_{fv} provides a closer approximation to the actual correction, the errror then being about 0.1% pessimistic for a 10:1 area ratio nozzle with $\theta = 15^O$.

4.3 Skin friction loss

The drag force acting on a nozzle expansion cone can be calculated as:

$$\text{Drag} = \int \frac{c_f}{2} \rho \, u^2 \, 2\pi \, r \, dx$$

where r is the local radius of the nozzle, and c_f is a skin friction coefficient. The skin friction coefficient can be calculated by compressible boundary layer methods which include the effects of the accelerating flow, but as a first approximation

$$\frac{c_f}{2} = 0.0256 \, Re_x^{-.2}$$

where Re_x is the Reynolds number based on distance down the flow. We can observe that c_f decreases as we move further downstream.

If we observe that

$$\dot{m} = \rho u \cdot \pi \, r^2 = \frac{P_o \, A_*}{c_*}$$

$$\text{Drag} = \int P_o \, A_* \cdot \frac{u}{c_*} \cdot \frac{c_f}{r} \, dx \quad .$$

Now u/c_* is a component of the thrust coefficient momentum balance, and like the thrust coefficient it increases only slowly as we go down the nozzle, with a value of about 0.7 at the throat increasing to about 1.65 for a 12:1 area ratio nozzle. The increase in u/c_* is actually rather greater than the decrease in skin friction coefficient, but it is probably close enough to write

$$\frac{\text{Drag}}{P_o \, A_*} = c_{fe} \frac{u_e}{c_*} \int \frac{f}{r} \, dx$$

where f is a non-dimensional factor accounting for the variation of $c_f \, u/c_*$ with x . We therefore see that if we define a skin friction loss efficiency

$$\eta_f = 1 - c_{fe} \int f \frac{dx}{r}$$

it will apply only to the velocity term in the thrust coefficient. Typically n_f is about 0.99.

4.4 Two-phase flow losses

Most particularly in the case of aluminized composite propellants (where up to 25% of the flow can be in the form of Al_2O_3 particles) the presence of solid particles in the flow induce flow losses both by retaining energy which should contribute to the thermodynamic expansion ("thermal lag") and by having a lower acceleration than the gas ("velocity lag"). If a fraction ξ of the exhaust is solid particles, the thrust is

$$\text{Thrust} = (1 - \xi)\,\dot{m}\,u_e + p_e\,A_e + \xi\,\dot{m}\,u_p$$

where u_p is the velocity of particles at the exit plane. If the velocity lag is ζ

$$T = (1 - \xi + \xi\zeta)\,\dot{m}\,u_e + p_e\,A_e \quad .$$

This simplified analysis suggests that the correction for two-phase flow should only be applied to the velocity term. However, it is a simplified analysis. If we examine in more detail what is happening within the nozzle we find that the drag of the solid particles will slow the velocity of the gas phase and increase its temperature, thereby reducing the Mach number of the gas. The thermal lag of the particles will also tend to add heat to the flow downstream. Since in the expansion cone at least thrust is generated only by the pressure of the gas, a reduction by Mach number represents a reduction in the whole of C_{fv} .

Part of the problem in assigning a correction for two-phase flow arises from the problems of calculating the effect accurately. A large part of the loss will occur upstream of the nozzle throat, and so will affect the characteristic velocity c^* also. It is significant that where exact formulations of two-phase flow problems have been carried out (e.g. Marble [3]) these have concentrated on calculating the reduction in specific impulse, combining C_f and c_* .

5 CONCLUSIONS

This memo has shown that the value for the ideal, "one-dimensional" vacuum thrust coefficient is the same when calculated by a momentum balance or by a

pressure distribution method. Investigation of the various losses suggests that the real thrust coefficient can be calculated from:

$$c_f = \eta_d \, \eta_t \left[\eta_f \, C_{fv} + (1 - \eta_f) \frac{P_e \, A_e}{P_o \, A_*} \right] - \frac{P_a \, A_e}{P_o \, A_*}$$

where
η_d is the divergence loss efficiency

η_t is the two-phase flow loss efficiency, and

η_f is the skin-friction loss efficiency.

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Nomenclature

A	conduit cross sectional area
A_o	cross sectional area at stagnation
A_e	nozzle exit area
A_*	throat area
a	local speed of sound
a_o	speed of sound at stagnation
a_*	speed of sound at the throat
c_*	characteristic velocity
F	thrust
$\dot{m}$	total mass flow rate
p	pressure
p_o	stagnation pressure
p_e	exit plane pressure
p_a	ambient pressure
R	gas constant per unit mass
r	local nozzle radius
T	absolute temperature
T_o	stagnation temperature
u	local gas velocity
u_e	exit plane gas velocity
u_*	throat gas velocity
γ	ratio of specific heats
ς	dimensionless velocity lag parameter
η_d	divergence loss efficiency
η_f	skin friction loss efficiency
η_t	two-phase flow loss efficiency
ξ	dimensionless thermal lag parameter
ρ	local gas density
ρ_o	stagnation gas density
ρ_*	gas density at throat

Non-dimensional numbers

C_f	thrust coefficienct
C_{fv}	thrust coefficient in vacuo
c_f	skin friction coefficient
c_{fe}	skin friction coefficient at exit plane
M	Mach number
M_o	nozzle entry Mach number
M_e	Mach number at exit plane
R_{ex}	Reynolds number based on distance down the flow

Fig. 1.
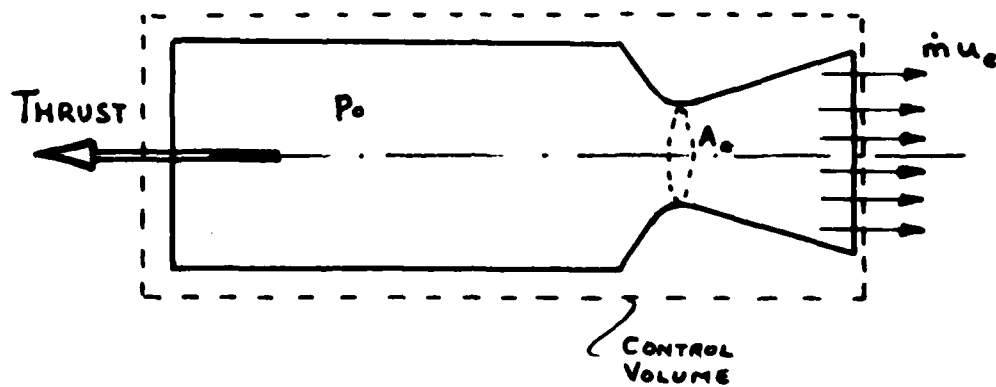
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Abstract

The thrust of a rocket motor is generally calculated from the stagnation pressure, the throat area, and a non-dimensional thrust coefficient dependent upon the nozzle expansion area ratio. A comparison is made between momentum balance and pressure integral methods of calculating the ideal vacuum thrust coefficient, and the results used to examine how corrections should be made for nozzle divergence, skin friction and two-phase flow to obtain the real thrust coefficient of the motor.

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